FINDING ALTERNATE OPTIMAL SOLUTIONS IN A TWO PERSON ZERO-SUM GAME WITH MS-EXCEL

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Abstract

Often in game theory there are ties in the value of the game by playing the strategies that produce the "best" or optimal result. In this paper, we provide several two person zero-sum game theory examples that produce alternate optimal solutions to playing the game. Further, we show how MS-Excel can be used to find these alternate optimal solutions.

Keywords: two person zero-sum game, mixed strategies, alternate optimal solutions, linear programming, movement diagram, saddle point method, MS-Excel

Introduction and Background to Mathematical Modeling

We teach a three course sequence of mathematical modeling to our students in our department at the Naval Postgraduate School. Although some students have had calculus, the course requirement for the sequence is only college algebra. We must build upon those algebra skills in covering modeling topics in discrete modeling, stochastic modeling, decision theory, and game theory modeling.

This paper discusses the third course with emphasis on the game theory portion. In that course, we use the 2004 Straffin text [5] and cover chapters 1, 2, 3, 4, 5, 7, 9, 10, 11, 12, 14, 15, 16, 19 and 26. As background in our first modeling course, we discuss linear programming - both the 2-variable graphical approach and the use of Excel's Solver with Simplex LP. In a lesson prior to game theory in the third course we conduct a two lesson thorough review of linear programming as decision making under certainty. We review both the graphical methods and using the Excel Solver to obtain linear programming solutions.

In the third course covering *total conflict models*, we spend a majority of the time on techniques and not the modeling process. We expect every student to complete a course project (of their own choosing) illustrating the modeling process and their model solution technique.

Here is a short list of some of the recent game theory projects:

- Game Theory with US and Non-State actors
- Game Theory in Cameroon-Nigeria dispute
- Game Theory in PMI and US military tasks
- COIN Game
- The Somali Pirates game
- US-Afghanistan Drug dilemma
- US-Afghanistan Regional Game
- US Coin Operations Game
- Dealing with Safe Havens as a Game
- IEDS and Counter-IEDS as a Game
- Game theory for Courses of Combat Action
- NFL Strike and Bargaining.

Prior to 2010, we usually restricted our instructor presentations to 2-person games where the players had 2 or 3 strategies each. This enabled manual calculations by instructors and students. However, reality dictates that most players have numerous strategies that they could employ. Having the facility to solve these games is important in their careers as they deal with courses of action that both they and their opponent may use. We use technology to assist us in the process. We use Excel because all our students will have Excel in their jobs after they graduate so we do not use specialized software.

The issue this paper exposes is mixed strategy alternate optimal solutions using linear programming. We will begin with an alternate optimal solution in pure strategy games where there are multiple methods to find these solutions.

We will use Straffin's convention [5] for the players. We will call the row player Rose and the column player as Colin. Thus, Rose is our row player and Colin is our column player for our games.

Pure Strategy Solutions

Most texts (Straffin [5]; Williams[6]; Giordano [4]) suggest using movement diagrams or the saddle point method (minimax=maximin) as the methods to examine a payoff matrix for pure strategy solutions for a specific class of zero-sum game problems. We quickly describe these methods.

Movement Diagrams

In a movement diagram, we draw vertical arrows from the smaller values in a row to the corresponding larger value in another row and then draw horizontal arrows from the other players smaller to larger values in columns. We tell the students to follow the arrows after they have drawn them in the payoff matrix. If following the arrows led into a point but no arrow leaves that point then that point is an equilibrium. We might find that multiple equilibria or no equilibrium may exist using this procedure. We illustrate with an example where Rose has four strategies {R1,R2,R3,R4} and Colin has four strategies {C1,C2,C3, C4}.

Example 1. Consider the following game between Rose and Colin where each has four strategies.

We draw the arrows as described. By following the arrows in Figure 1, we find the arrows all *point in* and no arrow departs from either R1C3 (2, -2) and R3C3 (2, -2). We have found two pure strategy solutions.

				Colin				
		C1		*C2*		*C3*		*C4*
Rose	*R1*	*(4-4)*	→	*(3,-3)*	→	*(2,-2)*	←	*(5,-5)*
	R2	*(-10,10)*	←	*(2-2)*	→	*(0,0)*	→	*(-1,1)*
	R3	*(7,-7)*	→	*(5,-5)*	→	*(2,-2)*	←	*(3,-3)*
	R4	*(0,0)*	←	*(8,-8)*	→	*(-4,4)*	→	*(-5,5)*

Figure 1. Movement Diagram for example 1.

Saddle Point Method

The following definition of a saddle point is taken from Straffin [5],

An outcome in a matrix game (with payoff to the row player) is called a saddle point if the entry at that outcome is both less than or equal to any entry in its row and greater than or equal to any entry in its column (p.9)

We illustrate the same example using this method using only the row player's payoffs.

We find a tie at R1C2=2, and R3C3=2 as before. The value of the game for Rose is this saddle point, *V=2*.

			Colin			Row minimums	Maximin
		C1	C2	C3	C4		
Rose	R1	4	3	2	5	2	2
	R2	-10	2	0	-1	-10	
	R3	7	5	2	3	2	2
	R4	0	8	-4	-5	-5	
	Colum maximum	7	8	2	5		
	Minimax			2			

Linear Programming Method

Linear programming is a method that can always be used in solving all zero sum games. We present a generic formulation for Rose's payoff matrix from Fox [1].

Maximize v_1-v_2

Subject to:

$$M_{1,1}x_1 + M_{2,1}x_2 + ... + M_{m,1}x_n - v_1 + v_2 \geq 0$$
$$M_{1,2}x_1 + M_{2,2}x_2 + ... + M_{2,m}x_n - v_1 + v_2 \geq 0$$
$$...$$
$$M_{1,m}x_1 + M_{2,m}x_2 + ... + M_{m,n}x_n - v_1 + v_2 \geq 0$$
$$x_1 + x_2 + ... + x_n = 1$$

Nonnegativity

where the weights x_i yield Rose strategies and the value of game, $V = v_1 - v_2$, is the value of the game to Rose. We use $V=v_1-v_2$ because the negatives in the payoff matrix might yield a negative value of the game, since all variables are non-negative this change in variables allows for negative values of the game [7].

Continuing our example with payoff matrix as shown in Figure 1, we formulate the linear program as follows:

Maximize v_1-v_2

$4x_1-10x_2+7x_3-v_1+v_2 \geq 0$
$3x_1+2x_2+5x_3+8x_4-v_1+v_2 \geq 0$
$2x_1+2x_3-4x_4-v_1+v_2 \geq 0$
$5x_1-x_2+3x_3-5x_4-v_1+v_2 \geq 0$
$x_1+x_2+x_3+x_4=1$
$x_i,v_j \geq 0 \quad i=1,2,3,4,\ j=1,2$

We use a template developed for class use to obtain a solution. We enter the number of rows and columns in cells F2:F3, Rose's payoff's in cells B7:E10, initialize the decision variables in cells B34:B55. We open the Solver dialog box, and press Solve.

We interpret the output in cells L2:L3, cells L18:L26 and cells B28:K28. The solution shown here is R1C3 with values 2 for Rose and -2 for Colin.

We save both the *Answer* and *Sensitivity* reports.

	A	B	C	D	E	F	G	H	I	J	K	L
1	This template will allow you to solve up to 10 strategies for each player in a two-person zero-sum game										Game Values	
2	Enter the number of Strategies for Rose					4					Rose	2
3	Enter the number of Strategies for Colin					4					Colin	-2
4												
5	Enter the payoff to the Row player only											
6	R/C	1	2	3	4	5	6	7	8	9	10	
7	1	4	3	2	5	0	0	0	0	0	0	
8	2	-10	2	0	-1	0	0	0	0	0	0	
9	3	7	5	2	3	0	0	0	0	0	0	
10	4	0	8	-4	-5	0	0	0	0	0	0	
11	5	0	0	0	0	0	0	0	0	0	1	
12	6	0	0	0	0	0	0	0	0	0	0	
13	7	0	0	0	0	0	0	0	0	0	0	
14	8	0	0	0	0	0	0	0	0	0	0	
15	9	0	0	0	0	0	0	0	0	0	0	
16	10	0	0	0	0	0	0	0	0	0	0	
17	R/C	1	2	3	4	5	6	7	8	9	10	Rose's strategies
18	1	4	3	2	5	0	0	0	0	0	0	1.0000
19	2	-10	2	0	-1	0	0	0	0	0	0	0.0000
20	3	7	5	2	3	0	0	0	0	0	0	0.0000
21	4	0	8	-4	-5	0	0	0	0	0	0	0.0000
22	5	0	0	0	0	0	0	0	0	0	0	0.0000
23	6	0	0	0	0	0	0	0	0	0	0	0.0000
24	7	0	0	0	0	0	0	0	0	0	0	0.0000
25	8	0	0	0	0	0	0	0	0	0	0	0.0000
26	9	0	0	0	0	0	0	0	0	0	0	0.0000
27	10	0	0	0	0	0	0	0	0	0	0	0.0000
28	Colin's	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
29	Strategies											

Iterations: 36 Subproblems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling

Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
B59		0	0

Variable Cells

Cell	Name	Original Value	Final Value	Integer
B32	x1	0.000000	1.000000	Contin
B33	x2	0.000000	0.000000	Contin
B34	x3	0.00000	0.00000	Contin
B35	x4	0.00000	0.00000	Contin
B36	x5	0.00000	0.00000	Contin
B37	x6	0.00000	0.00000	Contin
B38	x7	0.00000	0.00000	Contin
B39	x8	0.00000	0.00000	Contin
B40	x9	0.00000	0.00000	Contin
B41	x10	0.00000	0.00000	Contin
B42	v1	0.00000	2.00000	Contin
B43	v2	0.00000	0.00000	Contin
B44	y1	0.00000	0.00000	Contin
B45	y2	0.00000	0.00000	Contin
B46	y3	0.00000	1.00000	Contin
B47	y4	0.00000	0.00000	Contin
B48	y5	0.00000	0.00000	Contin
B49	y6	0.00000	0.00000	Contin
B50	y7	0.00000	0.00000	Contin
B51	y8	0.00000	0.00000	Contin
B52	y9	0.00000	0.00000	Contin
B53	y10	0.00000	0.00000	Contin
B54	v3	0.00000	0.00000	Contin
B55	v4	0.00000	2.00000	Contin

The Solver's main solution is x1=*1* corresponding to *R1* and y3=*1* corresponding to *C3* with a game value for Rose of $v_1-v_2=2-0=2$ and for Colin $v_3-v_4=0-2=-2$.

We examined the sensitivity report to look for the possibility of alternate optimal solutions to a zero-sum game. From the Sensitivity Report, we find the indicators of alternate optimal solutions.

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [LP Template for Zero Sum Game Theory feb 5 2014 4 x 4 paper 2.xlsm]ZERO SUM Template
Report Created: 12/1/2014 11:22:08 AM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B32	x1	1	0	0	1E+30	0
B33	x2	0	-2	0	2	1E+30
B34	x3	2.22045E-16	0	0	0	2
B35	x4	0	-6	0	6	1E+30
B36	x5	0	-2	0	2	1E+30
B37	x6	0	-2	0	2	1E+30
B38	x7	0	-2	0	2	1E+30
B39	x8	0	-2	0	2	1E+30
B40	x9	0	-2	0	2	1E+30
B41	x10	0	-2	0	2	1E+30
B42	v1	2	0	1	0	1
B43	v2	0	0	-1	0	1E+30
B44	y1	0	-2	0	2	1E+30
B45	y2	0	-1	0	1	1E+30
B46	y3	1	0	0	2	1
B47	y4	0	-3	0	3	1E+30
B48	y5	0	0	0	1E+30	2
B49	y6	0	0	0	1E+30	2
B50	y7	0	0	0	1E+30	2
B51	y8	0	0	0	1E+30	2
B52	y9	0	0	0	1E+30	2
B53	y10	0	0	0	1E+30	2
B54	v3	0	0	1	0	1E+30
B55	v4	2	0	-1	0	1E+30

Allowable decrease of 0.

Allowable increase of 0 and currently has value 0.

What about alternate solutions?

The indicators are found in the variable cells of the Sensitivity Report as highlighted here. Variable *x1* currently at *1* can decrease by 0 and *x3* can increase by *0*. This indicates possible alternate optimal solutions in Excel. So how do we find the alternate solution, if one exists?

First, we change the objective function from maximize v_1-v_2 to maximize x_3. We maximize x_3 because it is currently at value 0 and can replace x_1, which is currently at value 1. Then we add a new constraint that states the objective function remain at 2, v_1-v_2=2. After making those two modifications with the Solver's dialog box, we press Solve.

After making these two alterations to the formulation, we can obtain the alternate solution, *x3=1*, *y3=1* *V=2*, corresponding to strategies *R3C3* obtaining a value of *2*.

Since there are no other variables with increases or decreases allowable to 0 in the original or this solution's sensitivity report we have found all the alternate solutions.

A	B	C	D	E	F	G	H	I	J	K	L
This template will allow you to solve up to 10 strategies for each player in a two-person zero-sum game										Game Values	
Enter the number of Strategies for Rose					4					Rose	2
Enter the number of Strategies for Colin					4					Colin	-2
Enter the payoff to the Row player only											
R/C	1	2	3	4	5	6	7	8	9	10	
1	4	3	2	5	0	0	0	0	0	0	
2	-10	2	0	-1	0	0	0	0	0	0	
3	7	5	2	3	0	0	0	0	0	0	
4	0	8	-4	-5	0	0	0	0	0	0	
5	0	0	0	0	0	0	0	0	0	1	
6	0	0	0	0	0	0	0	0	0	0	
7	0	0	0	0	0	0	0	0	0	0	
8	0	0	0	0	0	0	0	0	0	0	
9	0	0	0	0	0	0	0	0	0	0	
10	0	0	0	0	0	0	0	0	0	0	
R/C	1	2	3	4	5	6	7	8	9	10	Rose's strategies
1	4	3	2	5	0	0	0	0	0	0	0.0000
2	-10	2	0	-1	0	0	0	0	0	0	0.0000
3	7	5	2	3	0	0	0	0	0	0	1.0000
4	0	8	-4	-5	0	0	0	0	0	0	0.0000
5	0	0	0	0	0	0	0	0	0	0	0.0000
6	0	0	0	0	0	0	0	0	0	0	0.0000
7	0	0	0	0	0	0	0	0	0	0	0.0000
8	0	0	0	0	0	0	0	0	0	0	0.0000
9	0	0	0	0	0	0	0	0	0	0	0.0000
10	0	0	0	0	0	0	0	0	0	0	0.0000
Colin's	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
Strategies											

Mixed Solutions

The typical solution methods for mixed solution, algebra, or the method of oddments do not support finding alternate solutions. Thus, we turn to linear programming using the same formulation and template as before.

Example 2. *4 x 5* Game Payoff Matrix[5]

In the following game (Straffin [5] exercise 2 chapter 2) Rose has four strategies and Colin has five strategies. Using the techniques from Straffin's solution, we use dominating strategies and then oddments to find as the solution: $V=v_1-v_2=4/3$ when $x_1=2/3$, $x_2=0$, $x_3=1/3$ corresponding to play R1 with probability 2/3, to never play R2, and to play R3 with probability 1/3 while playing C3 with probability 2/3, to play C5 with probability 1/3, and to never playing strategies C1, C2, or C4.

			Colin			
		C1	*C2*	*C3*	*C4*	*C5*
Rose	*R1*	*1*	*1*	*1*	*2*	*2*
	R2	*2*	*1*	*1*	*1*	*2*
	R3	*2*	*2*	*1*	*1*	*1*
	R4	*2*	*2*	*2*	*1*	*0*

We employ linear programming to solve this more complex problem quickly and to look for possible alternate optimal solutions.

Maximize v_1-v_2

Subject to

$x_1+2x_2+2x_3+2x_4-v_1+v_2\geq 0$

$x_1+x_2+2x_3+2x_4-v_1+v_2\geq 0$

$x_1+x_2+x_3+2x_4-v_1+v_2\geq 0$

$2x_1+x_2+x_3+x_4-v_1+v_2\geq 0$

$2x_1+2x_2+2x_3-v_1+v_2\geq 0$

$x_1+x_2+x_3+x_4=1$

$x_i,v_j\geq 0$ for $i=1,2,3,4,\ j=1,2$

Using our linear programming template and Solving on Excel, we obtain the following solution.

	A	B	C	D	E	F	G	H	I	J	K	L
2	Enter the number of Strategies for Rose					4					Rose	1.333333333
3	Enter the number of Strategies for Colin					5					Colin	-1.333333333
4												
5	Enter the payoff to the Row player only											
6	R/C	1	2	3	4	5	6	7	8	9	10	
7	1	1	1	1	2	2	0	0	0	0	0	
8	2	2	1	1	1	2	0	0	0	0	0	
9	3	2	2	1	1	1	0	0	0	0	0	
10	4	2	2	2	1	0	0	0	0	0	0	
11	5	0	0	0	0	0	0	0	0	0	1	
12	6	0	0	0	0	0	0	0	0	0	0	
13	7	0	0	0	0	0	0	0	0	0	0	
14	8	0	0	0	0	0	0	0	0	0	0	
15	9	0	0	0	0	0	0	0	0	0	0	
16	10	0	0	0	0	0	0	0	0	0	0	
17	R/C	1	2	3	4	5	6	7	8	9	10	Rose's strategies
18	1	1	1	1	2	2	0	0	0	0	0	0.3333
19	2	2	1	1	1	2	0	0	0	0	0	0.3333
20	3	2	2	1	1	1	0	0	0	0	0	0.0000
21	4	2	2	2	1	0	0	0	0	0	0	0.3333
22	5	0	0	0	0	0	0	0	0	0	0	0.0000
23	6	0	0	0	0	0	0	0	0	0	0	0.0000
24	7	0	0	0	0	0	0	0	0	0	0	0.0000
25	8	0	0	0	0	0	0	0	0	0	0	0.0000
26	9	0	0	0	0	0	0	0	0	0	0	0.0000
27	10	0	0	0	0	0	0	0	0	0	0	0.0000
28	Colin's	0.0000	0.3333	0.3333	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000	
29	Strategies											

We note that, using linear programming, our solution is to play *R1* with probability 1/3, R2 with probability 1/3, and R4 with probability 1/3 while playing *C2* with probability 1/3, *C3* with probability 1/3, and *C5* with probability 1/3 yielding a solution to the game of $V=v_1-v_2=4/3$.

We note that this solution differs for the solution provided by Straffin [5]. Our applying linear programming yielded an alternate optimal solution to the previous method from Straffin.

We examine the sensitivity report.

Further analysis of the sensitivity report shows that x_3 might replace either x_2 or x_4 as a basic variable in the solution and y_1 might replace either affect y_2 or y_3. Our queues are as before.

In a systemic way, we look for the other solutions. Not only can we obtain the solution as described above but also *V=v1-v2=4/3* when *x1=2/3, x2=0, x3=1/3* corresponding to play *R1* with probability 2/3 never play *R2* and play *R3* with probability 1/3 while playing *C3* with probability 2/3 and *C5* with probability 1/3 and never playing strategies *C1*, *C2*, or *C4*, but we can find the following solutions as well - all yielding a value of the game of *V=4/3, R2=1/3, R3=2/3, R1=R4=0; C1=C4=0, C2=C3=C5=1/3.*

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [LP Template for Zero Sum Game Theory feb 5 2014 4 x 5 alt op example 2 mixed Dec 2.xlsm]ZERO SUM Template
Report Created: 12/2/2014 11:49:02 AM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B32	x1	0.333333333	0	1	1E+30	1
B33	x2	0.333333333	0	0	0	1E+30
B34	x3	0	0	0	0	0
B35	x4	0.333333333	0	0	1E+30	0
B36	x5	0	1	0	1E+30	1
B37	x6	0	1	0	1E+30	1
B38	x7	0	1	0	1E+30	1
B39	x8	0	1	0	1E+30	1
B40	x9	0	1	0	1E+30	1
B41	x10	0	1	0	1E+30	1
B42	v1	2	0	0	1E+30	0
B43	v2	0.666666667	0	0	1	0
B44	y1	1.11022E-16	0	0	0	0.333333333
B45	y2	0.333333333	0	0	0	0
B46	y3	0.333333333	0	0	0.5	0
B47	y4	0	0.333333333	0	1E+30	0.333333333
B48	y5	0.333333333	0	0	0.5	1
B49	y6	0	0	0	1.333333333	1E+30
B50	y7	0	0	0	1.333333333	1E+30
B51	y8	0	0	0	1.333333333	1E+30
B52	y9	0	0	0	1.333333333	1E+30
B53	y10	0	0	0	1.333333333	1E+30
B54	v3	0	0	0	1E+30	0
B55	v4	1.333333333	0	0	1E+30	0

Ultimately, we have found 12 alternate solutions to this game and suspect that there might be more. We would not have searched at all if we had not used linear programming and noticed the conditions for alternate optimal solution existed.

R1	*R2*	*R3*	*R4*	*C1*	*C2*	*C3*	*C4*	*C5*	*V*
2/3	*0*	*1/3*	*0*	*0*	*0*	*2/3*	*0*	*1/3*	*4/3*
2/3	*0*	*1/3*	*0*	*0*	*1/3*	*1/3*	*0*	*1/3*	*4/3*
2/3	*0*	*1/3*	*0*	*0*	*1*	*0*	*0*	*0*	*4/3*
2/3	*0*	*1/3*	*0*	*0*	*0*	*1*	*0*	*0*	*4/3*
2/3	*0*	*1/3*	*0*	*0*	*0*	*0*	*0*	*1*	*4/3*
2/3	*0*	*1/3*	*0*	*1*	*0*	*0*	*0*	*0*	*4/3*
1/3	*2/3*	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*4/3*
1/3	*1/3*	*0*	*1/3*	*0*	*0*	*0*	*0*	*1*	*4/3*
2/3	*0*	*0*	*1/3*	*0*	*0*	*1*	*0*	*0*	*4/3*
0	*2/3*	*1/3*	*0*	*0*	*0*	*1*	*0*	*0*	*4/3*
0	*0*	*2/3*	*1/3*	*0*	*0*	*1*	*0*	*0*	*4/3*
0	*2/3*	*0*	*1/3*	*0*	*0*	*2/3*	*0*	*1/3*	*4/3*

Conclusion

Why do we care about alternate solutions? As we tell our students the mathematics in game theory do not dictate how we play the game but provides quantitative assessment of how the game would be played repetitively if both players are rational players attempting to obtain the best solution they can. Further when looking for "good" courses of action, which is what the best possible results that the strategy selections tell us, it is good to know if more pure or mixed courses of action exist for the decision makers. This allows for more flexibility by the decision maker. In our examples, had we not used linear programming, we would have accepted the single answer for the only strategies to play.

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